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On a transcendental equation related to Painlevé III, and its discrete forms

A Ramani†, B Grammaticos‡, T Tamizhmani‡‖ and K M Tamizhmani§

† CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France
‡ GMPIB, Université Paris VII, Tour 24-14, 5^e étage, 75251 Paris, France
§ Department of Mathematics, Pondicherry University, Kalapet, Pondicherry 605014, India

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Abstract. We examine the canonical forms of Painlevé equations and argue that the equation for P_{III} in which one parameter is taken to be equal to zero should be considered as a canonical form different from the standard P_{III}. Our argument is based on the fact that the value of this parameter cannot be modified through auto-Bäcklund transformations. We investigate the possible discrete forms of this equation and produce two of them. One is of a difference type, where the independent variable enters linearly, while the second one is of q type where the independent variable enters in a multiplicative way. The properties of these discrete equations are also studied.

1. Introduction

The discovery of the Painlevé transcendents [1] is one of the major successes in the theory of integrable systems. These new functions are defined from the solution of integrable second-order differential equations (ODEs). Their explicit integration was presented much later, by Ablowitz and Segur [2]. The complete classification of second-order ODEs possessing the Painlevé property was initiated by Painlevé himself, and completed by Gambier [3]. Common belief is that there are 50 such equations and that six of these define new transcendents. Gambier presented a minimal list of 24 equations and clearly stated that when all possible transformations are considered the resulting forms can be counted well into the hundreds. The pseudocanonical list of 50 is essentially based on Ince's book [4] which is the standard reference on the subject. Recently, Cosgrove [5] has challenged this common belief (in a work that remains, alas, unpublished) and presented his own canonical lists of 74, 81 and even 120 equations.

Given this situation one can ask whether there are just six Painlevé transcendents, defined through the second-order ODEs examined by Painlevé and Gambier. Since the functional forms of the equations are given it is clear that the only freedom that is left is related to the parameters that appear in five of the Painlevé equations. Now, it is well known that the Painlevé equations have auto-Bäcklund and Schlesinger transformations which relate the solutions of a Painlevé equation for a given set of parameters to the solution for some other set of parameters [6]. Thus, in general, no special values of the parameters exist, since they can be modified through the auto-Bäcklund transformations. However, there exist situations where some parameters cannot be modified by auto-Bäcklund transformations. This is the case for two of the Painlevé

‖ Permanent address: Department of Mathematics, Kanchi Mamunivar Centre for Postgraduate Studies, Pondicherry, India.

equations, namely P_{III} and P_V. For the latter it is well known [3] that there exists a special case where the equation can be transformed to P_{III}. Thus no new transcendent is introduced in this case. However, in the former case the situation is not as simple. In this paper we shall study the one-parameter P_{III} equation and show that it has properties different from those of the full P_{III}. We shall also present two different discretizations. One is a discrete Painlevé equation of difference type, while the other is a multiplicative, q-discrete, one. For both, we present the nonlinear and bilinear forms. The equations obtained through the appropriate Miura transforms are also given.

2. The continuous one-parameter P_{III} equation

Let us consider the P_{III} equation

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w} \tag{2.1}$$

where the prime ($'$) indicates derivation with respect to the independent variable t. It is clear, given the form of (2.1), that only two parameters are significant: the remaining two can be put to 1 through a scaling of the dependent and the independent variables. This is possible, of course, only if their value is not zero. Thus the cases where these parameters vanish may constitute new canonical forms associated to P_{III}. Let us examine (2.1) in more detail.

The general form of P_{III} corresponds to (2.1) with $\gamma\delta \neq 0$. The standard normalization is $\gamma = 1, \delta = -1$.

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(\alpha w^2 + \beta) + w^3 - \frac{1}{w}. \tag{2.2}$$

The auto-Bäcklund transformation of (2.1) in this case is

$$\begin{aligned} w_{\alpha\pm2,\beta+2} &= \pm\frac{1}{w}\frac{t(\frac{w'}{w} \pm w + \frac{1}{w}) - 1 - \beta}{t(\frac{w'}{w} \pm w + \frac{1}{w}) + 1 \pm \alpha} \\ w_{\alpha\pm2,\beta-2} &= \mp\frac{1}{w}\frac{t(\frac{w'}{w} \pm w - \frac{1}{w}) - 1 + \beta}{t(\frac{w'}{w} \pm w - \frac{1}{w}) + 1 \pm \alpha} \end{aligned} \tag{2.3}$$

where $w_{\alpha+p,\beta+q}$ denotes the solution of (2.2) where α and β are translated by p and q, and where we have only dropped the indices for the basic solution, i.e. $w \equiv w_{\alpha,\beta}$. From (2.3) we see that the values $\alpha = 0$ and/or $\beta = 0$ do not play any special role since they can be modified through the application of the auto-Bäcklund transformation.

Let us assume now that $\gamma = 0$, while $\alpha\delta \neq 0$. This is the case we are going to focus on here. Clearly, we can choose the scaling of the dependent and independent variable so as to have $\alpha = 1$, $\delta = -1$, and the remaining parameter is now called η. Thus the one-parameter P_{III} equation takes the form

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(w^2 + \eta) - \frac{1}{w}. \tag{2.4}$$

This equation is different from the standard form (2.2) since the missing term cannot be restored. It has only one free parameter and is different from the other one-parameter Painlevé equation, P_{II}. (We often refer to equation (2.4) as 'lame-P_{III}', since it lacks one of the two degrees of freedom of the full P_{III}.) Equation (2.4) has many, but not all, properties of the full P_{III}. In particular, its auto-Bäcklund transformation

$$w_{\eta\pm2} = t\frac{1 \pm w'}{w^2} - \frac{\eta \pm 1}{w} \tag{2.5}$$

allows one to modify the value of η. (Thus $\eta = 0$ does *not* play a special role.) In (2.5) we have used the same notational convention as for (2.3). Two of the properties of P_{III} are still missing in (2.4). It does not lead to P_{II} through coalescence but only to P_I. Indeed, putting $w = \epsilon^{-5} - \epsilon^{-1}u$, $t = -2\epsilon^{-15}(1 + \epsilon^8 z)$, $\eta = -3\epsilon^{-10}$, we find, at the limit $\epsilon \to 0$, Painlevé I, $u'' = 6u^2 + z$, where the prime ($'$) now denotes the derivative with respect to z. Moreover, equation (2.4) does not possess special solutions obtained through a reduction to a Riccati equation, which shows it is essentially different from P_{II}. A property of the one-parameter P_{III} that is not so well known is the one concerning its Miura transformation. The Miura transformation can be expressed as a system

$$\phi = t\frac{w' + 1}{w} \tag{2.6a}$$

$$w\phi' = w^2 - \phi + \eta + 1. \tag{2.6b}$$

Eliminating ϕ between (2.6*a*), (2.6*b*), and the derivative of the former gives the one-parameter P_{III} equation (2.4) for w. On the other hand, eliminating w between (2.6*a*), (2.6*b*) and the derivative of the latter leads to a second-order, second-degree equation for ϕ:

$$(\phi'' + 2)^2 = \frac{\phi^2}{t^2}(\phi'^2 + 4\phi - 4\eta - 4). \tag{2.7}$$

Equation (2.7) is precisely equation SD-III.A identified by Cosgrove and Scoufis [7] in the study of second-degree Painlevé equations.

By putting more parameters to zero one can obtain further canonical forms of P_{III}. The case $\gamma = \delta = 0$, $\alpha\beta \neq 0$, is a zero-parameter P_{III} (since α and β can be scaled, for instance, to 1 and -1, respectively). This equation can be transformed through $w \to w^2$, $t \to t^2$ to a P_{III} with $\alpha = \beta = 0$ and $\gamma\delta \neq 0$, whereupon the values of α, β can be modified through the use of auto-Bäcklund transformations. Finally, the case $\gamma = \alpha = 0$ (or, equivalently, $\beta = \delta = 0$) is a case where P_{III} can be reduced to an autonomous equation which can be integrated in terms of elliptic functions.

2.1. Discrete equations from the auto-Bäcklund transformations

Before proceeding further let us recall the discrete equations obtained from the auto-Bäcklund transformations of the two- and one-parameter P_{III}. The general theory of this construction was presented in [8]. The main idea is that one uses the auto-Bäcklund transformation in order to construct solutions corresponding to adjacent values of some parameter and then eliminates the first derivative between these relations. The resulting equation is a mapping where the independent variable is the parameter of the continuous equation. The discrete equation associated to the two-parameter P_{III} obtained from (2.3) in a straightforward way is

$$\frac{\alpha + \beta + 2}{1 - ww_{\alpha+2,\beta+2}} + \frac{\alpha + \beta - 2}{1 - ww_{\alpha-2,\beta-2}} = 2t\left(w + \frac{1}{w}\right) + 2\alpha \tag{2.8}$$

where we recall that w stands for $w_{\alpha,\beta}$, a discrete Painlevé equation known as the alternate d-P_{II} and which was studied in great detail in [9]. In that paper, we presented the Miura transformation of alternate d-P_{II} in the form of a four-point mapping. It turns out that it is possible to obtain a simpler Miura transformation which leads to a three-point mapping. The derivation is based on the bilinear formalism and the self-dual description of alternate d-P_{II} we presented in [9], and here we will give just the results without details. We introduce new variables u and x which are *not* themselves solutions of (2.2) but are related to the latter and live on the same lattice but at different points. We have the system

$$x_{\alpha-2,\beta} - t = \frac{(\alpha+\beta-2)w}{ww_{\alpha-2,\beta-2} - 1}$$
$$u_{\alpha-1,\beta+1} + \frac{\mu}{2} = \frac{x_{\alpha-2,\beta}+t}{w} \tag{2.9}$$
$$u_{\alpha-1,\beta+1} - \frac{\mu}{2} = -(x_{\alpha,\beta+2}+t)w$$

where $\mu = \beta - \alpha + 2$ is a constant along the evolution. By eliminating u and x one recovers the alternate d-P_{II} equation (2.8). On the other hand, eliminating w leads to a system for u and x which is the Miura transformation of the alternate d-P_{II}:

$$(x_{\alpha,\beta+2}+t)(x_{\alpha-2,\beta}+t) = \frac{\mu^2}{4} - u^2_{\alpha-1,\beta+1} \tag{2.10a}$$

$$u_{\alpha-3,\beta-1} + u_{\alpha-1,\beta+1} = (\alpha+\beta-2)\frac{t+x_{\alpha-2,\beta}}{t-x_{\alpha-2,\beta}}. \tag{2.10b}$$

It turns out that if one tries to eliminate u between (2.10*a*), its downshift (by two units in both α and β) and (2.10*b*), one does obtain a three-point mapping for x but which is quadratic in $x_{\alpha,\beta+2}$ and $x_{\alpha-4,\beta-2}$. This equation was also presented in [9] where we remarked that its discrete derivative is just the four-point mapping mentioned above. On the other hand, it is possible to eliminate x between (2.10*a*), (2.10*b*) and its upshift (by two units in both α and β) and get a three-point mapping for u which is now homographic for $u_{\alpha-3,\beta-1}$ and $u_{\alpha+1,\beta+3}$, namely

$$\frac{(u_{\alpha-3,\beta-1}+u_{\alpha-1,\beta+1}+\alpha+\beta-2)(u_{\alpha-1,\beta+1}+u_{\alpha+1,\beta+3}+\alpha+\beta+2)}{(u_{\alpha-3,\beta-1}+u_{\alpha-1,\beta+1})(u_{\alpha-1,\beta+1}+u_{\alpha+1,\beta+3})} = \frac{16t^2}{\mu^2 - 4u^2_{\alpha-1,\beta+1}}. \tag{2.11}$$

This equation is a limit of the d-P_V equation which has been recently proposed [10]. Equation (2.11) constitutes a discrete form for the equation P_{34} [4], and was first proposed in [11].

The discrete equation obtained from the auto-Bäcklund transformation (2.5) is

$$w_{\eta+2} + w_{\eta-2} = \frac{2t - 2\eta w}{w^2} \tag{2.12}$$

which is a discrete form of P_I [8]. As we see in what follows, this equation can also go to the zero-parameter P_{III} ($\gamma = \delta = 0$). Its Miura transformations have been studied in [12]. Indeed, defining again a new variable u, which is not a solution of (2.4), through $u_{\eta+1} = ww_{\eta+2} - (\eta+1)$, we obtain the equation

$$(u_{\eta-1}+u_{\eta+1})(u_{\eta+1}+u_{\eta+3}) = \frac{4t^2}{u_{\eta+1}+\eta+1} \tag{2.13}$$

which was identified in [13] as another discrete form of P_I.

3. The discrete, one-parameter difference-P_{III} equation

The gist of the above analysis is that the continuous P_{III} has two distinct canonical forms which can be distinguished by the number of effective parameters entering the equation. We now turn to the investigation of the discrete analogues [14] of these equations and, in particular, that of the one-parameter, since the discrete forms of the full P_{III} are well known. As we have shown in previous work [14, 15], the discrete Painlevé equations appear under two distinct forms depending on whether the independent variable enters in an additive or multiplicative way. In the first case we have difference (sometimes referred to as δ) equations, while in the second case we have q equations. The equations we are going to derive must go over to the one-parameter P_{III} at the continuous limit. Let us start with the discrete, difference equation. As we have shown in [16], the full discrete P_{III} can be represented by the system

$$x + \bar{x} = \frac{\tilde{z}y + a}{y^2 - c^2} \tag{3.1a}$$

$$y + \underset{\sim}{y} = \frac{zx + b}{x^2 - d^2} \tag{3.1b}$$

where $z = z_0 + \kappa n$, $\tilde{z} = z + \kappa/2$ and $\bar{x}$, $\underset{\sim}{y}$ stand for $x(n+1)$ and $y(n-1)$, respectively. The standard normalization of (3.1) is $c = d = 1$. Equation d-P_{III} has been studied in great detail in [17] where we have given its self-dual description through the analysis of the geometry of the evolution and its Schlesinger transformations. Equation (3.1) can be obtained from the auto-Bäcklund transformations of P_V. The discrete equivalent of the one-parameter P_{III}, equation (2.4), can be obtained from (3.1) by taking $c = 0$. We thus find

$$x + \bar{x} = \frac{\tilde{z}y + a}{y^2} \tag{3.2a}$$

$$y + \underset{\sim}{y} = \frac{zx + b}{x^2 - d^2}. \tag{3.2b}$$

By taking $y = \frac{s}{2x} + \frac{sx'-x}{4x^2}\epsilon + \mathcal{O}(\epsilon^2)$, $a = \epsilon^2/32$, $b = -\eta\epsilon^2/16$, $d = \epsilon/4$, where we have taken $\kappa = \epsilon$, we find at the continuous limit $\epsilon \to 0$, so that $z \to s$, the equation

$$x'' = \frac{x'^2}{x} - \frac{x'}{s} + \frac{1}{4s^2}(x^2 + \eta s) - \frac{1}{4x} \tag{3.3}$$

where ($'$) denotes the derivative with respect to s. Equation (3.3) is the one-parameter P_{III} although in noncanonical form. Putting $x = tw$ and $s = t^2$ we recover exactly equation (2.4). Equation (3.2) has, just like its continuous counterpart, some but not all of the properties of the full d-P_{III}. In particular, it does not have special-function-type solutions and does not degenerate towards d-P_{II} through coalescence, but only to d-P_I. As a matter of fact, taking $d = 0$ allows one to introduce a scaling such that $a = b$, in which case (3.2*b*) is just the downshift of (3.2*a*) (by half a step in z). The resulting equation is a d-P_I: with the proper identification of the parameters it is simply equation (2.12). On the other hand, if we consider the continuous limit that leads to (3.3) it is easy to retrace the steps with $d = 0$. The result is then the same as equation (3.3) with the $1/x$ term missing. Thus, the $d = 0$ case, which coincides with equation (2.12), has a continuous limit to the zero-parameter P_{III} ($\gamma = \delta = 0$). We must make clear, however, that these two continuous limits of (2.12) exist in different sectors. If we consider equation (2.12) as resulting from the one-parameter d-P_{III}, then we can follow the different behaviours of y in the two continuous limits. In the P_I sector, y is the same as x, just upshifted by half a step, while in the P_{III} sector y is roughly equal to $z/2x$.

3.1. Relation to P_{III} and bilinearization

It is interesting to show that one can obtain equation (3.2) from the auto-Bäcklund transformations of a continuous equation. It turns out that the same Schlesinger transformations of the continuous P_{III} which, as we have seen, led to the alternate d-P_{II}, equation (2.8), also give the discrete one-parameter P_{III}, equation (3.2). We introduce $y = w_{\alpha-2,\beta+2}$, $\underset{\sim}{y} = w_{\alpha-2,\beta-2}$ and, using equation (2.3), we find

$$y + \underset{\sim}{y} = \frac{2\beta(t(w' - w^2) - (\alpha-1)w) + 2(\alpha-2)t}{(t(w' - w^2) - (\alpha-1)w)^2 - t^2}. \tag{3.4a}$$

This is indeed identical to equation (3.2b) provided we define $x = (t(w' - w^2) - (\alpha - 1)w)$, which, incidentally, is exactly the quantity x introduced in equation (2.9), and we identify $z = 2\beta$, $d = t$, $b = 2(\alpha - 2)t$. Next, using equation (2.3) and the differential equation for P_{III}, equation (2.2), we can prove that x defined above satisfies two relations in terms of y and $\underline{y}$, namely $x = t(y' - 1)/y^2 + (\beta + 1)/y$ and $x = t(-\underline{y}' - 1)/\underline{y}^2 + (\beta - 1)/\underline{y}$. Thus 'upshifting' the second relation (i.e. writing it for $\beta + 4$) and adding to the first one we find

$$x + \bar{x} = -\frac{2t}{y^2} + \frac{2\beta + 4}{y} \tag{3.4b}$$

which is just (3.2a) with $a = -2t$, and $2\beta + 4$ is precisely $\tilde{z}$, since a full step κ of $z = 2\beta$ is $\kappa = 8$.

Since the one-parameter discrete P_{III} and the alternate discrete P_{II} are both obtained from the Schlesinger transformations of the continuous P_{III} it makes sense to compare them, in particular, as motions in the (α, β) plane. The variable y is essentially the same as w, just shifted to the point $(\alpha - 2, \beta + 2)$, but note that in (2.8) the motion is by two units in both α and β (the dual equation where α and β vary in opposite directions also exists), while in (3.2) the motion is by four units in β only (the dual motion in α only also exists). Thus the two motions are at an angle of 45° to each other. The geometry of alternate d-P_{II} and its Schlesinger transformations have been described in detail in [9]. Using these results of [9] it is straightforward to obtain the bilinearization of the one-parameter d-P_{III}. We start by introducing the two-parameter τ-function which exists on points of the two-dimensional lattice $\tau_{\alpha+2m,\beta+2p}$, such that $m + p$ is odd. We now define w as

$$w_{\alpha,\beta} = \frac{\tau_{\alpha,\beta+2}\tau_{\alpha,\beta-2}}{\tau_{\alpha+2,\beta}\tau_{\alpha-2,\beta}}. \tag{3.5}$$

The quantity w (or equivalently, $y = w_{\alpha-2,\beta+2}$) exists at points $(\alpha + 2m, \beta + 2p)$ such that $m + p$ is even. The quantity x, on the other hand, lives at the same points on the lattice as the τ and is defined by

$$x_{\alpha-2,\beta} = t - \frac{\tau_{\alpha,\beta+2}\tau_{\alpha-4,\beta-2}}{\tau^2_{\alpha-2,\beta}} = \frac{\tau_{\alpha,\beta-2}\tau_{\alpha-4,\beta+2}}{\tau^2_{\alpha-2,\beta}} - t. \tag{3.6}$$

A first bilinear equation is obtained by equating the two expressions for x:

$$\tau_{\alpha,\beta+2}\tau_{\alpha-4,\beta-2} + \tau_{\alpha,\beta-2}\tau_{\alpha-4,\beta+2} = 2t\tau^2_{\alpha-2,\beta}. \tag{3.7}$$

Substituting the expressions of x and y in terms of τ into equation (3.2) we obtain multilinear equations which can be separated into the bilinear system

$$\tau_{\alpha-4,\beta-2}\tau_{\alpha+2,\beta} - \tau_{\alpha,\beta+2}\tau_{\alpha-2,\beta-4} = (\beta + \alpha - 2)\tau_{\alpha,\beta-2}\tau_{\alpha-2,\beta} \tag{3.8a}$$

$$\tau_{\alpha+2,\beta}\tau_{\alpha-4,\beta+2} + \tau_{\alpha,\beta-2}\tau_{\alpha-2,\beta+4} = (\beta - \alpha + 2)\tau_{\alpha,\beta+2}\tau_{\alpha-2,\beta}. \tag{3.8b}$$

Equations (3.7) and (3.8) are an overdetermined but compatible system, and constitute the bilinearization of the one-parameter d-P_{III}, equation (3.2), as well as the alternate d-P_{II}, equation (2.8), as can be easily checked.

3.2. *The Miura transformed equations*

Just as we have done in the continuous case, we shall introduce the Miura transformation of the one-parameter d-P_{III}. The equation we are seeking is one relating the quantity u introduced in (2.9) and a new variable which we shall denote by v. Our guide, as usual, will be the bilinear formalism. Using (2.9) and the expressions for x, w in terms of τ we find $u_{\alpha-1,\beta+1} + \mu/2 = \tau_{\alpha-4,\beta+2}\tau_{\alpha+2,\beta}/\tau_{\alpha,\beta+2}\tau_{\alpha-2,\beta}$. Next, we introduce v which, in terms of the τ-functions can be written $v_{\alpha-1,\beta-1} = \tau_{\alpha-4,\beta-2}\tau_{\alpha+2,\beta}/\tau_{\alpha,\beta-2}\tau_{\alpha-2,\beta}$. It is now staightforward to show that $v_{\alpha-1,\beta-1}+u_{\alpha-1,\beta+1}+\mu/2 = 2t/w_{\alpha,\beta}$ and $v_{\alpha-1,\beta+3}+u_{\alpha-1,\beta+1}+\mu/2 = 2t/w_{\alpha-2,\beta+2}$, using the same convention as for the one-parameter d-P_{III}, equation (3.2). Multiplying the two relations we find a denominator $ww_{\alpha-2,\beta+2}$ which, using (3.8*a*), can be expressed in terms of $u_{\alpha-1,\beta+1}$ only. The result is a first equation relating $v_{\alpha-1,\beta-1}$, $v_{\alpha-1,\beta+3}$ and $u_{\alpha-1,\beta+1}$ which reads

$$(v_{\alpha-1,\beta-1} + u_{\alpha-1,\beta+1} + \mu/2)(v_{\alpha-1,\beta+3} + u_{\alpha-1,\beta+1} + \mu/2) = 4t^2 \frac{\mu/2 + u_{\alpha-1,\beta+1}}{\mu/2 - u_{\alpha-1,\beta+1}}. \tag{3.9}$$

It is clear from the form of (3.9) that we can obtain a much simpler expression if we shift u. Thus redefining $u \to u - \mu/2$ and dropping the first index which is always $\alpha - 1$, we have

$$(v_{\beta-1} + u_{\beta+1})(v_{\beta+3} + u_{\beta+1}) = 4t^2 \frac{u_{\beta+1}}{\beta + 1 - u_{\beta+1} - (\alpha - 1)} \tag{3.10a}$$

where $\beta + 1$ is the value at $u_{\alpha-1,\beta+1}$ of the independent variable while $\alpha - 1$ is the constant parameter along this evolution, and we have used the value of $\mu = \beta - \alpha + 2$. The second equation relating u and v can be obtained along similar lines, using the fact that $v_{\alpha-1,\beta-1} + u_{\alpha-1,\beta-3} = 2t/w_{\alpha-2,\beta-2}$. The final result is

$$(v_{\beta-1} + u_{\beta-3})(v_{\beta-1} + u_{\beta+1}) = -4t^2 \frac{v_{\beta-1}}{\beta - 1 - v_{\beta-1} + \alpha - 1}. \tag{3.10b}$$

Here the value of the independent variable at $v_{\alpha-1,\beta-1}$ is indeed $\beta - 1$, shifted down by a half-step from (3.10*a*). System (3.10) is the Miura transformation of the one-parameter d-P_{III} equation. Its continuous limit can be obtained by putting $4t^2 = \epsilon$, $\beta = 4s/\epsilon$ and $v_{\alpha-1,\beta-1} = -u_{\alpha-1,\beta+1} + \epsilon x/2s + \mathcal{O}(\epsilon^2)$. At the limit $\epsilon \to 0$ we obtain the system

$$u + \alpha - 2 = -s\frac{2x' + 1}{x} \tag{3.11a}$$

$$2xu' = \frac{x^2}{s} - u \tag{3.11b}$$

where the (′) denotes the derivative with respect to s. This is the analogue of system (2.6). Note that up to a trivial normalization, x here is essentially the quantity $x_{\alpha,\beta+2}$ appearing in equation (2.9). Eliminating u we obtain, for x, a one-parameter d-P_{III} in the noncanonical form (3.3), with α instead of η, while eliminating x leads to a second-degree equation for $\phi = u + \alpha - 2$ similar to (2.7).

One remark is in order at this point. In [9] we have presented the solutions of alternate d-P_{II} in terms of special functions (Bessel or, equivalently, discrete-Airy functions). These solutions are written in terms of τ-functions which are Casorati determinants, the size of which is determined by $\alpha - \beta$. This size is, therefore, constant along the evolution defined by equation (2.8): they are what are called ‘lattice’ solutions. At the continuous limit these solutions go over to solutions of continuous P_{II} in terms of fixed-size Wronskian determinants of Airy functions. Alternate d-P_{II} also has solutions where the size of the determinants is determined by $\alpha+\beta$ and keeps varying along the evolution: such solutions are called ‘molecule’ solutions in the same terminology. These solutions do not possess a continuous limit. The very same τ-functions also define solutions for equation (3.2) but here the evolution is along β for fixed α and, therefore, the size of the determinants always varies along the evolution. Thus the one-parameter d-P_{III} has two families of ‘molecule’ solutions but no lattice-type solutions. In particular, as we stated above, it does not have special-function-type solutions obtained through a reduction to a Riccati equation (which would be elementary lattice-type solutions, in terms of determinants the size of which is just one).

4. The one-parameter q-$\mathrm{P_{III}}$ equation

We turn now to the q-discrete form of the one-parameter $\mathrm{P_{III}}$. The starting point is the full q-$\mathrm{P_{III}}$, first obtained in [18]:

$$x\bar{x} = \frac{aqy + \lambda q^2}{y(y+d)} \tag{4.1a}$$

$$y\underline{y} = \frac{bqx + q^2}{x(x+c)} \tag{4.1b}$$

where $q = q_0\lambda^n$. (We must point out here that the equation known as standard q-$\mathrm{P_{III}}$ is, as shown by Jimbo and Sakai [19], a q-discrete form of $\mathrm{P_{VI}}$ when its full freedom is considered, and is thus not quite appropriate for our purposes.) The standard normalization of (4.1) is $c = d = -1$ and, in fact, the continuous limit of (4.1) is the full $\mathrm{P_{III}}$. The one parameter q-$\mathrm{P_{III}}$ can be obtained from (4.1) in a straightforward way by taking $c = 0$. We gauge x through $x \to qx/b$, take $b^2 = d$ without loss of generality since it corresponds to a scaling of y, and redefine the independent variable through $z = \lambda dq/a$. We thus find the canonical form of this q equation:

$$x\bar{x} = \frac{1 + y/z}{y(1 + y/d)} \tag{4.2a}$$

$$y\underline{y} = d\frac{x+1}{x^2}. \tag{4.2b}$$

The continuous limit of equation (4.2) is obtained by putting: $y = \epsilon^{-1}w/t$, $x = -\epsilon t/w + \mathcal{O}(\epsilon^2)$, $\lambda = \mathrm{e}^{-2\epsilon}$, $z = -2\epsilon^{-3}/t^2$ and $d = 1 + \eta\epsilon$, resulting in the one-parameter continuous $\mathrm{P_{III}}$, equation (2.4).

4.1. Bilinearization and Miura transformations

At this point it is interesting to study, in more detail, equation (4.2). We start by deriving the Schlesinger transformations, which correspond to the changes of the parameter d. We shall denote these changes by the symbol hat, i.e. $\hat{d} = \lambda d$. We first introduce an auxiliary variable v through the relations

$$\bar{x}yv = xy\hat{v} = 1. \tag{4.3}$$

Using (4.3) and the one-parameter q-$\mathrm{P_{III}}$ we can now establish the equations for the evolution along the d direction:

$$v\hat{v} = \frac{1 + y/d}{y(1 + y/z)} \tag{4.4a}$$

$$y\underset{\hat{}}{y} = z\frac{v+1}{v^2}. \tag{4.4b}$$

We remark that this equation has the same form as (4.2), provided one replaces x by v and interchanges z and d. Thus the one-parameter q-$\mathrm{P_{III}}$ is self-dual. Before proceeding further, let us give the bilinearization of equation (4.2). We start with the ansatze: $x_{z,d} = \frac{\tau\tau_{z/\lambda,d}}{\tau_{z,\lambda d}\tau_{z/\lambda,d/\lambda}}$, $v_{z,d} = \frac{\tau\tau_{z,d/\lambda}}{\tau_{\lambda z,d}\tau_{z/\lambda,d/\lambda}}$ and $y_{z,d} = \frac{\tau_{\lambda z,\lambda d}\tau_{z/\lambda,d/\lambda}}{\tau^2} = z(\frac{\tau_{\lambda z,d}\tau_{z/\lambda,d}}{\tau^2} - 1) = d(\frac{\tau_{z,\lambda d}\tau_{z,d/\lambda}}{\tau^2} - 1)$, where τ stands for $\tau_{z,d}$. Note that though for simplicity we gave to x, v and y the indices (z, d), this is really only strictly true for y, the actual values of the independent variable and of the parameter being in fact $(z/\sqrt{\lambda}, d)$ for x and $(z, d/\sqrt{\lambda})$ for v. Using these ansätze one can obtain two bilinear equations:

$$\tau_{\lambda z,\lambda d}\tau_{z/\lambda,d/\lambda} = z(\tau_{\lambda z,d}\tau_{z/\lambda,d} - \tau^2) = d(\tau_{z,\lambda d}\tau_{z,d/\lambda} - \tau^2) \tag{4.5}$$

and verify that (4.2*a*) is identically satisfied. A third bilinear equation is obtained from equation (4.2*b*) and we give it in two forms equivalent up to (4.5):

$$\tau_{\lambda z,\lambda^2 d}\tau_{z/\lambda,d/\lambda} = z(\tau\tau_{z,\lambda d} + \tau_{z/\lambda,d}\tau_{\lambda z,\lambda d}) \tag{4.6a}$$

$$\tau_{\lambda^2 z,\lambda d}\tau_{z/\lambda,d/\lambda} = d(\tau\tau_{\lambda z,d} + \tau_{z,d/\lambda}\tau_{\lambda z,\lambda d}). \tag{4.6b}$$

Just as we have done for the continuous and discrete one-parameter d-P_{III}, here we shall introduce the Miura transformation of equations (4.2) and (4.4). We start by introducing the auxiliary variable u defined in terms of the τ-functions as $u_{z,d} = \frac{\tau_{z/\lambda,d}\tau_{z,d/\lambda}}{\tau\tau_{z/\lambda,d/\lambda}}$, the actual values of the independent variable and of the parameter being, in fact, $(z/\sqrt{\lambda}, d/\sqrt{\lambda})$. The Miura transformations involving u read

$$u_{z,d} = x_{z,d}(1 + y_{z,d}/d) = v_{z,d}(1 + y_{z,d}/z) \tag{4.7a}$$

$$u_{\lambda z,\lambda d} = x_{\lambda z,d}(1 + y_{z,d}/d) = v_{z,\lambda d}(1 + y_{z,d}/z). \tag{4.7b}$$

Using equations (4.7) together with (4.3) we can easily recover (4.2) and (4.4). On the other hand we can also obtain their Miura transformations by eliminating y and one of the variables x or v. For instance, the elimination of y and x, leads to the Miura transformations of (4.2):

$$\left(\frac{u_{z,d}}{v_{z,d}} - 1\right)\left(\frac{u_{\lambda z,d}}{v_{z,d}} - 1\right) = \frac{1}{z}\frac{v_{z,d} + 1}{v_{z,d}^2} \tag{4.8a}$$

$$\left(\frac{u_{z,d}}{v_{z,d}} - 1\right)\left(\frac{u_{z,d}}{v_{z/\lambda,d}} - 1\right) = \frac{d}{z}(1 + u_{z,d}). \tag{4.8b}$$

This equation can assume a 'nicer' form if we invert v. Thus with $v \to 1/v$ we obtain

$$(u_{z,d}v_{z,d} - 1)(u_{\lambda z,d}v_{z,d} - 1) = \frac{1}{z}v_{z,d}(v_{z,d} + 1) \tag{4.9a}$$

$$(u_{z,d}v_{z,d} - 1)(u_{z,d}v_{z/\lambda,d} - 1) = \frac{d}{z}(1 + u_{z,d}). \tag{4.9b}$$

Similarly, the Miura transformation of (4.4) is obtained by elimination of y and v resulting in (with $x \to 1/x$):

$$(u_{z,d}x_{z,d} - 1)(u_{z,\lambda d}x_{z,d} - 1) = \frac{1}{d}x_{z,d}(x_{z,d} + 1) \tag{4.10a}$$

$$(u_{z,d}x_{z,d} - 1)(u_{z,d}x_{z,d/\lambda} - 1) = \frac{z}{d}(1 + u_{z,d}) \tag{4.10b}$$

where, like in (4.4) d is the independent variable and z is a parameter. The continuous limit of (4.9) and (4.10) is a second-degree equation of Cosgrove type. Putting $u_{z,d} = -1 - \epsilon\phi/2$, $v_{z,d} = -1 + \epsilon\phi/2 + \mathcal{O}(\epsilon^2)$ in (4.9) and choosing the same ansatze for λ, z and d as for the continuous limit of (4.2) we obtain exactly equation (2.7) for ϕ.

4.2. The oblique equations

We now turn to the derivation of the equation that relates x and v. In geometrical terms the motions along z and d define evolutions along two orthogonal axes while the evolution we are looking for now is a diagonal one, at 45°, where both z and d advance by λ. Thus $z = z_0\lambda^n$ and $d = d_0\lambda^n$, where n is the number of steps in *both* the z and d directions. Using again, (4.2) and (4.3), we readily find

$$v_{z/\lambda,d}v_{z,\lambda d} = \frac{1}{d(x_{z,d} + 1)} \tag{4.11a}$$

$$x_{z,d}x_{\lambda z,\lambda d} = \frac{1}{z(v_{z,\lambda d} + 1)}. \tag{4.11b}$$

This equation was first derived in this form in [18] where it was shown to be equivalent to some other equation, namely

$$(zx_{z,d}x_{\lambda z,\lambda d}-1)(zx_{z,d}x_{z/\lambda,d/\lambda}/\lambda-1)=\frac{1}{d(x_{z,d}+1)} \tag{4.12}$$

obtained in [13] as a limit of q-$\mathrm{P_V}$. At the continuous limit it goes over, as expected, to $\mathrm{P_{II}}$. Indeed, taking $x_{z,d}=-2(1+\epsilon w)$, $v_{z,d}=-2(1-\epsilon w)+\mathcal{O}(\epsilon^2)$, $z_0=-\frac{1}{4}$, $d_0=-(1+\epsilon^3\mu)/4$ and $\lambda=1+\epsilon^3/2$ (which means that $\lambda^n=1+\epsilon^2t/2$ with $t=n\epsilon$), we obtain, at the limit $\epsilon\to0$, the equation $w''=2w^3+tw-2\mu-\frac{1}{2}$: i.e. exactly Painlevé II in canonical form. Equation (4.11) describes an equation along a diagonal direction involving x. It is also possible to define a diagonal evolution which involves y and u. Using the Miura transformations (4.7) introduced above we can easily find

$$u_{z,d}u_{\lambda z,\lambda d}=\frac{(1+y_{z,d}/d)(1+y_{z,d}/z)}{y_{z,d}} \tag{4.13a}$$

$$y_{z,d}y_{z/\lambda,d/\lambda}=(1+u_{z,d})zd/\lambda. \tag{4.13b}$$

From the form of (4.13*b*) it is clear that we can eliminate u altogether and obtain an equation for y only. We thus find

$$(y_{z,d}y_{\lambda z,\lambda d}-\lambda zd)(y_{z,d}y_{z/\lambda,d/\lambda}-zd/\lambda)=zd\frac{(y_{z,d}+d)(y_{z,d}+z)}{y_{z,d}}. \tag{4.14}$$

This equation, up to a trivial gauge, is part of the q-$\mathrm{P_V}$ family discussed in [13], although not explicitly analysed there. We can show that (4.14) is a discrete form of $\mathrm{P_{34}}$. Indeed putting $y_{z,d}=(1+\epsilon^2(w+t/2))/4$ and taking the same ansatze for λ, z and d as for the continuous limit of (4.11) we obtain, at the limit $\epsilon\to0$, the equation $w''=w'^2/2w-4w^2-tw-\mu^2/2w$, i.e. $\mathrm{P_{34}}$ in almost canonical form.

Equation (4.11) being a discrete $\mathrm{P_{II}}$ possesses solutions in terms of q-discrete special functions. We find readily that if $d_0=z_0$, x and v can be expressed as $x_{z,d}=-2A_{z,d}/A_{z/\lambda,d/\lambda}$, $v_{z,\lambda d}=-A_{z,d}/2zA_{\lambda z,\lambda d}$, where A is the solution of the q-discrete Airy equation $A_{z/\lambda,d/\lambda}-2A_{z,d}-4zA_{\lambda z,\lambda d}=0$. At the continuous limit A satisfies the Airy equation $A''+tA/2=0$, and the corresponding $u=A'/A$ indeed satisfies $\mathrm{P_{II}}$ with $\mu=0$. Higher solutions of this type exist whenever $d_0/z_0=\lambda^m$, where m is an integer, and can be obtained using the Schlesinger transformations given by equations (4.2) and (4.4). This means that the same objects can be considered as solutions of (4.2) and (4.4). However, since the solutions are expressed as ratios of determinants in A the size of which grows with the evolution steps along the orthogonal axes ('bar' or 'hat', independently) these solutions of the one-parameter q-$\mathrm{P_{III}}$ are again of molecule type.

Equation (4.11) is not the only transverse equation we can obtain from (4.2), (4.3). Indeed, we can consider a motion where z and d advance in opposite directions, i.e. when d becomes λd, z becomes z/λ. The equation for x and v is slightly more complicated in this case. We find

$$\frac{(x_{\lambda z,d}-v_{\lambda z,d})(x_{\lambda z,d}-v_{z,\lambda d})}{(x_{\lambda z,d}-dv_{\lambda z,d}/\lambda z)(x_{\lambda z,d}-dv_{z,\lambda d}/z)}=\frac{x_{\lambda z,d}+1}{dx^2_{\lambda z,d}} \tag{4.15a}$$

$$\frac{(v_{z,\lambda d}-x_{z,\lambda d})(v_{z,\lambda d}-x_{\lambda z,d})}{(v_{z,\lambda d}-zx_{z,\lambda d}/\lambda d)(v_{z,\lambda d}-zx_{\lambda z,d}/d)}=\frac{v_{z,\lambda d}+1}{zv^2_{z,\lambda d}}. \tag{4.15b}$$

This equation can assume a more familiar form if we introduce the variable change $v\to1/v$. We thus obtain

$$\frac{(x_{\lambda z,d}v_{\lambda z,d}-d/\lambda z)(x_{\lambda z,d}v_{z,\lambda d}-d/z)}{(x_{\lambda z,d}v_{\lambda z,d}-1)(x_{\lambda z,d}v_{z,\lambda d}-1)}=\frac{dx^2_{\lambda z,d}}{x_{\lambda z,d}+1} \tag{4.16a}$$

$$\frac{(v_{z,\lambda d}x_{z,\lambda d} - \lambda d/z)(v_{z,\lambda d}x_{\lambda z,d} - d/z)}{(v_{z,\lambda d}x_{z,\lambda d} - 1)(v_{z,\lambda d}x_{\lambda z,d} - 1)} = \frac{\lambda d^2}{z v_{z,\lambda d}(v_{z,\lambda d} + 1)} \tag{4.16b}$$

where the independent variable is now d/z and the only parameter is zd. Equation (4.16) is a limit of the asymmetric q-P_{VI} which has been recently derived [10]. It is, of course, a very particular limit since out of the six parameters of asymmetric q-P_{VI} only one survives here. We can remark here that equations (4.11) and (4.15) look completely different. Contrary to the case of equations (4.2)–(4.4) the motion along the diagonal direction is not self-dual. Thus although the space evolution of the one-parameter q-P_{III} and its Schlesinger transformations is a two-dimensional one it does not have the symmetries of either a B_2 or an A_2 Weyl group, which would have been the case in a perfectly self-dual situation.

5. Conclusion

We can now sum up our findings. In this paper we have studied an often overlooked canonical form of the Painlevé III equation. We have shown that this one-parameter P_{III} has properties different than those of the full P_{III} (and also than those of the other one-parameter Painlevé equation, P_{II}). We have presented two different discretizations of this one-parameter P_{III}. The first is a discrete Painlevé equation of difference type while the second is a q-Painlevé equation. We have shown that the difference equation, as expected, is perfectly self-dual. Moreover, the q equation is also self-dual when one considers the motion along the two main directions: evolution in z or changes of the parameter d through Schlesinger transformations. However, the motion in diagonal directions when the original independent variable z and the parameter d both change, either together or in opposition, is *not* self-dual. This makes the geometrical description of this equation all the more challenging.

The method used in this paper was to study an equation, its difference form and its q form in parallel. We consider that this is the right approach to the study of discrete systems since it establishes a parallel with continuous systems and explores the two aspects of discreteness: the additive and the multiplicative. We intend to use this mode of complete investigation in future work on discrete systems.

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